Solution of the Initial Value Problems for Systems of Non-linear Differential-Algebraic Equations Via Adomian Decomposition Method

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Abstract— In This paper we use Adomian decomposition method to solve the initial value problems for systems of the differential-algebraic equations that can be transformed to system of differential equations. The Adomian decomposition method (ADM) is a powerful method which considers the approximate solution of a non-linear equation as an infinite series which usually converges to the exact solution. This method is proposed to solve some first-order differential-algebraic equations. It is shown that the series solutions converge to the exact solution for each problem. It is observed that the method is particularly suited for initial value problems for systems of non-linear differential-algebraic equations.

Keyword— Differential equations, Differential-Algebraic Equations, Initial Value Problems, Adomian Decomposition Method

1 INTRODUCTION

The system of differential-algebraic equations indicates a system that consists of ordinary differential equations coupled with purely algebraic equations; in other words, differential-algebraic equations are everywhere singular implicit ordinary differential equations. In other define differential-algebraic equations are systems of differential equations where the unknown functions satisfy additional algebraic equations [Griepentrog E. and et al., 1992] [Wenhai C. and Zhengyi L.2004]presented An Algorithm for Adomian Decomposition Method, Applied Mathematics and Computation. [S. K. and A. A. 2011] solve differential algebraic equations using a multiquadric approximation scheme.[M. N. and et.al.2006] they is applied Adomian Decomposition Method (ADM) to typical oscillation equations (Duffing and Van der Pol equations). [G. N. and et.al. 2016] observed that the method of (ADM) is particularly suited for initial value problems with oscillatory and exponential solutions.

Consider the system of differential-algebraic equations that consist of the first order non-linear ordinary differential equations: $y'(t) = f\,(t, y(t), x(t))$ (1.1a)

together with the nonlinear algebraic equation

$$g(t, y(t), x(t)) = 0 \qquad (1.2b)$$

If we differentiate the algebraic constraint (1.1) with respect to t, one can get:

$$g_y(t, y(t), x(t))y'(t) + g_x(t, y(t), x(t))x'(t) = -g_t(t, y(t), x(t))$$

if g_x is nonsingular, the system given by eq.(1.3) has index one. The number of differentiation steps required in this procedure is the index [Brenan K. and et al. 1989].

2.The Adomian decomposition method [Wnhai C. and Zhengyi L., 2004]:

The Adomian decomposition method has been applied to solve problems in physics, biology and chemical reactions. Recently, there has been a great deal of interest in applying Adomian's decomposition technique for solving a wide class of nonlinear equations, including algebraic, differential, partial-differential, differential-delay and integro-differential equations

For nonlinear models, the method has shown reliable results in supplying analytical approximation that converges very rapidly.

The key of the method is to decompose the nonlinear term in the equations into a series of polynomials

$$\sum_{n=1}^{\infty} A_n ,$$

where A_n are the so-called Adomian polynomials.

The basic principles of the Adomian decomposition methods for solving differential equations by considering the general equation:

$$Fu = g \qquad (2.8)$$

where F represents a general nonlinear differential operator involving both linear and nonlinear terms, the linear term is decomposed in to $L+R$, where L is easily invertible and R is the remainder of the linear operator, Nu represents the non-linear terms. For convenience, L may be taken as the highest order derivative If the highest derivative that appeared in eq.(2.8) is n, then $L=\frac{d^n}{dt^n}$. Thus eq.(2.8) may be written as:

$$Lu+Ru+Nu=g$$

By solving Lu from the above equation, one can have:

$$Lu=g-Ru-Nu \tag{2.9}$$

Since L is an invertible operator, eq.(2.9) becomes

$$L^{-1}Lu=L^{-1}g-L^{-1}Ru-L^{-1}Nu \tag{2.10}$$

For example, if L is a second-order operator, then L^{-1} is a twofold integration operator, and

$L^{-1}Lu(t)=u(t)-u(0)-tu'(0)$, in this case eq.(2.10) for u yields:

$$u(t)=u(0)+u'(0)t+L^{-1}g-L^{-1}Ru-L^{-1}Nu$$

Therefore, u can be presented as a series:

$$u=\sum_{n=0}^{\infty}u_n, \tag{2.11}$$

where $u_0=\sum_{i=0}^{n}u^i(0)\frac{t^i}{i!}+L^{-1}g$, and u_n, n=1,2,... are to be determined. The nonlinear term Nu will be decomposed by the infinite series of Adomian polynomials,

$$Nu=\sum_{n=0}^{\infty}A_n, \tag{2.12}$$

where A_n's are said to be Adomain polynomials and can be obtained by writing:

$$v(\lambda)=\sum_{n=0}^{\infty}\lambda^n u_n \tag{2.13}$$

and

$$N(v(\lambda))=\sum_{n=0}^{\infty}\lambda^n A_n \tag{2.14}$$

Here λ is a parameter introduced for convenience. From eq.(2.13) and eq.(2.14), one can deduce that:

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N(v(\lambda))\right]_{\lambda=0}, \text{n=0,.1,...} \tag{2.15}$$

Now, by substituting eq.(2.11) and eq.(2.12) into eq.(2.10), one can obtain:

$$\sum_{n=0}^{\infty}u_n=u_0-L^{-1}R\sum_{n=0}^{\infty}u_n-L^{-1}\sum_{n=0}^{\infty}A_n.$$

Consequently, we can write:

$$u_0(t)=\sum_{i=0}^{n}u^i(0)\frac{t^i}{i!}+L^{-1}g,$$

$$u_1(t)=-L^{-1}Ru_0-L^{-1}A_0$$

$$\vdots$$

$$u_{n+1}(t)=-L^{-1}Ru_n-L^{-1}A_n$$

From the above equations, all of u_n are calculable and hence we substitute these values into eq.(2.11). Since the series converges and does so very rapidly, the n-term partial sum $\varphi_n=\sum_{i=0}^{n-1}u_i$ can serve as an approximated solution of above equations.

To illustrate this method, we consider the following example.

Example (2.1):

Consider the initial value problem that consists of the first order nonlinear ordinary differential equation:

$$y'(t)=-y(t)x(t),\ t\geq 0 \tag{2.16.a}$$

together with nonlinear algebraic equation:

$$e^{-y(t)x(t)}-\sin y(t)x(t)+\sin(1)-e^{-1}=0,\ t\geq 0 \tag{2.16.b}$$

and with initial conditions

$$y(0)=1 \tag{2.16.c}$$

This example is constructed such that:

$$y(t)=1-t,\ x(t)=\frac{1}{1-t}$$

To solve this example, we use Adomian decomposition method. To do this we must transform the above system of differential-algebraic equations to system of differential equations. To do this we must convert the algebraic equation given by eq.(2.16.c) into an equivalent differential equation. To do this, we differentiate eq.(2.16.b) with respect to x to get:

$$-e^{-y(t)x(t)}[y'(t)x(t)+x'(t)y(t)]=\cos[x(t)y(t)][y'(t)x(t)+x'(t)y(t)]$$

By substituting eq.(2.16.a) into the above equation and after simple calculated one can get:

$$x'(t)=x^2(t) \tag{2.16.d}$$

After that, we set t=0 and y(0)=1 into eq.(2.16.b) to get:

$$e^{-x(0)}-\sin[x(0)]+\sin(1)-e^{-1}=0$$

which has the solution

$$x(0)=1 \tag{2.16.e}$$

Therefore the initial value problem give by eq.(2.16) reduces to the initial value problem that consist of two of the first order nonlinear ordinary differential equations

$$y'(t)=-y(t)x(t) \tag{2.17.a}$$

$$x'(t) = x^2(t) \tag{2.17.b}$$

together with the initial conditions:

y(0)=1 (2.17.c)

x(0)=1 (2.17.d)

From eq.(2.17.b) one can notice that:

R(x(t))=0, $N(x(t)) = x^2(t)$, g(x(t))=0

to get x0(t), by using Adomian decomposition method such that:

$x_0(t) = x(0) + L^{-1}g(x(t))$,

one can have:

$x_0(t) = 1$.

to fined $x_1(t)$, $x_2(t)$, …, $x_n(t)$ by using the following equation.

$$\sum_{n=0}^{\infty} x_n(t) = x_0(t) - L^{-1}R\sum_{n=0}^{\infty} x_n(t) - L^{-1}\sum_{n=0}^{\infty} A_n.$$

But R(x(t))=0, therefore

$x_1(t) = L^{-1}A_0$.

where A_0 defined as previous in eq.(2.15) then

A_0=1

respectively

$$L^{-1}A_0 = \int_0^t dx = t$$

thus

$x_1(t)$=t

also

$x_2(t) = L^{-1}A_1$. where

$$A_1 = \frac{1}{1!}\left(\frac{d}{d\lambda}N(v(\lambda))\right)\bigg|_{\lambda=0}$$ such that

$$v(\lambda) = \lambda^0 x_0(t) + \lambda^1 x_1(t)$$

therefore

$$A_1 = \frac{1}{1!}\left(\frac{d}{d\lambda}\left(\lambda^0 x_0(t) + \lambda^1 x_1(t)\right)^2\right)\bigg|_{\lambda=0}$$

thus

$$x_2(t) = \int_0^t 2x_1(t)dx = t^2$$

for simplicity we truncate in this term to get:

$x(t) = 1 + t + t^2$

on the other hand by substituting x(t)in eq.(2.17.a) one can have:

$y'(t) = -y(t) - ty(t) - t^2 y(t)$

respectively

$R(y(t)) = -y(t) - ty(t) - t^2 y(t)$, $N(x(t)) = 0$, g(x(t))=0

similar to the previous one can have:

$$y(t) = 1 - t - 0.1666t^3 + 0.4583t^4 + 0.1666t^5 + 0.0555t^6$$

the following table represent the comparison between the exact solutions and the approximate solution by Adomain decomposition

Table (2.1) represents the exact and the approximated solutions of example (2.3)

i	t_i	$y_i(t_i)$	y_i	$x_i(t_i)$	x_i
1	0.05	0.95	0.95	1.0526	1.0525
2	0.1	0.9	0.8999	1.1111	1.11
3	0.15	0.85	0.8497	1.1765	1.1725
4	0.2	0.8	0.7995	1.25	1.24
5	0.25	0.75	0.7494	1.3333	1.3325
6	0.3	0.7	0.7997	1.4286	1.42
7	0.35	0.65	0.6507	1.5385	1.5325
8	0.4	0.6	0.603	1.6667	1.6600
9	0.45	0.55	0.5571	1.8182	1.8125
10	0.5	0.5	0.5039	2	1.95

Conclusion:

In this paper, the standard Adomian decomposition method is used to solve some nonlinear differential-algebraic equations, this method gives analytical solution in series form which converges rapidly. The reliability and the reduction in the size of computational work is certainly a sign of a wider applicability of the method.

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